

An Algorithm on Harmonious Labeling of Fan and Friendship Graphs

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Abstract

In this paper, we present an algorithm for enumerating all possible harmonious labelings of a fan graph f_n for any given number n of vertices and a friendship graph F_n for any given number n of triangle components. Our numerical scheme is based on Dushyant Tanna's construction proof of an injective labeling function $f:V(G) \rightarrow \{0, 1, 2, \dots, q-1\}$ for a graph G with q edges such that its induced function $f^*:E(G) \rightarrow \{0, 1, 2, \dots, q-1\}$ defined by $f^*(uv) = f(u) + f(v) \pmod{q}$ is a bijection. Here, $V(G)$ and $E(G)$, respectively denote the vertex set and edge set of graph G , which may either be a fan or friendship graph. We, then implement our algorithm using the Fortran programming language and plot the results using a command-line Gnuplot program. Finally, we present a pseudocode of our numerical scheme and some computational examples.

Keywords: algorithm, harmonious labeling, fan graphs, friendship graphs, Fortran programing

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Introduction

Graph theory is an interesting field of Mathematics because of its diversified applications. Interest in graph theory and its applications have grown exponentially in the last century due to the rise of information and communications technology (ICT) age. Graphs play a vital role in the establishment of physical network systems (e.g., electrical circuit, transportation network, telephone network) and in explaining ecological systems, sociological relationships, and flow of control in a computer system. This leads to a wide variety of interesting research topics in graph theory, which includes domination in graph, decomposition of graph, graph coloring, and graph labeling.

In this work, we are interested in graph labeling, This refers to the assignment of nonnegative integers to the vertices or edges of the graph, or in some cases both the vertices and edges, subject to certain conditions (Gallian, 2013). In particular, we focus on harmonious labeling of graphs. Such labeling of graphs has numerous applications in social networking, rare probability event, solving typical algorithms of coloring problems, and transmission networks (Tanna, 2013). As an illustration, one can consider a network of transmitting stations, each of which must be able to communicate with other linked stations. If there are n stations in the network, the total available bandwidth must be divided into n channels and each station x is assigned a channel number $\lambda(x)$ from Z_+ . When stations x and y communicate, they must use channel $\lambda(x) + \lambda(y)$. Hence, for proper transmission of data, each of these channel numbers must be assigned to exactly one link – requiring harmonious labeling (Sloane, 1980). The same goes for computers or devices that are connected to the internet, an IP address (Internet Protocol) is assigned to it and tells us where it is located, anywhere in the world. These IP addresses are also used for communication between devices in a network. Assigning such unique addresses results in a harmonious labelling problem. Harmonious graphs are also of interest because they lead to *modular* versions of various combinatorial problems. Harmonious labeling of the *friendship graph*, for example, may be regarded as a modular generalization of the Langford-Skolem problem (Skolem, 1957).

This paper is organized as follows. We begin with some preliminary concepts on harmonious labeling of fan and friendship graphs, largely based on Dushyant Tanna's proof (Tanna, 2013). From this, we

write our algorithm for enumerating all possible harmonious labelings of a fan graph f_n for any given number n of vertices and a friendship graph F_n for any given number n of triangle components.. We, then implement our algorithm using the Fortran programming language (Markus, 2012) and plot the results using a command-line Gnuplot program (Williams and Kelly, 2015). Finally, we present a pseudocode of our numerical scheme and some computational examples.

Preliminary Concepts and Related Works

This preliminary section fixes notations and states some definitions and theorems needed to build our algorithm in the succeeding section. For a more comprehensive discussion, we refer the reader to J. A. Bondy and S. R. Murty’s book entitled “Graph Theory with Applications” and Dushyant Tanna’s paper entitled “Harmonious Labeling of Certain Graphs”.

On Graphs

A *graph* G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G .

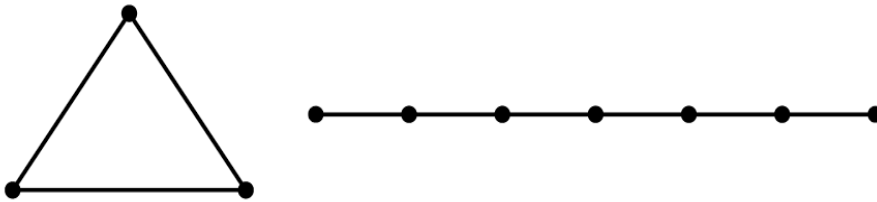


Figure 1. Examples of Graphs: Triangle (left) and Path P_7 (right).

A triangle, for example, has 3 vertices, each of which are adjacent to each other, resulting in 3 edges. A path P_n with n vertices is a sequence $P_n = v_1 e_1 v_2 e_2 v_3 \dots e_{n-1} v_n$, whose terms are alternately vertices and edges such that, for $1 \leq i \leq n-1$, the ends of edge e_i are vertices v_{i-1} and v_i (See Figure 1).

In this paper, we shall focus on fan and friendship graph. A fan graph, denoted by f_n ($n \geq 2$) is obtained by joining all n vertices of P_n to another vertex called the center. This graph contains $n+1$ vertices

and $2n-1$ edges, i.e., $f_n = P_n + K_1$. On the other hand, a friendship graph, denoted by F_n , is a graph consisting of n triangles with one common vertex (See Figure 2).

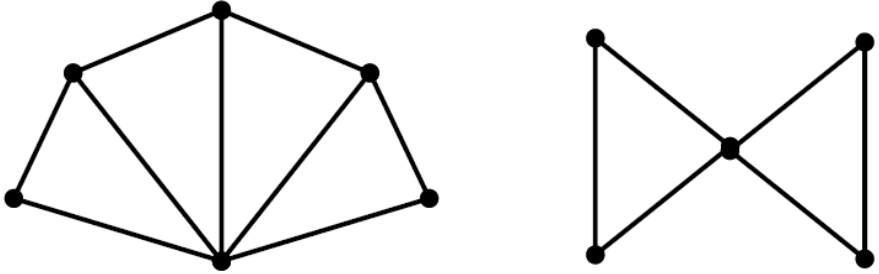


Figure 2. An illustration of a fan graph f_5 (left) and fan graph F_2 (right).

On Graph Labeling

A graph G is said to be *labeled* if each vertex and/or edge is assigned some number (usually a natural number) called *label*. In particular, a graph labeling requires: (1) a set of numbers from which the vertex labels are chosen; (2) a rule that assigns a value to each edge; and (3) a condition that vertex labels and/or edges labels must satisfy (Lawas, 2016). The earliest works on graph labelings were introduced by A. Rosa in his paper “On Certain Valuations of the Vertices of a Graph” (Rosa, 1967); and R.L. Graham and N.J.A. Sloane’s “On Additive Bases and Harmonious Graphs” (Graham and Sloane, 1980). Rosa introduced the assignment of vertex and edge labels as a β -valuation of a graph (Rosa, 1967).

In this paper, we are interested in graph labelings that are said to be harmonious. To be precise, a graph G with q edges is said have a *harmonious labeling* if there exists an injective function $f: V(G) \rightarrow \{0, 1, 2, \dots, q-1\}$ such that its induced function $f^*: E(G) \rightarrow \{0, 1, 2, \dots, q-1\}$, defined by $f^*(uv) = (f(u) + f(v)) \pmod{q}$ is bijective. Such graph, which admits harmonious labeling is called a harmonious graph.

The following theorems presents a harmonious labeling of both fan and friendship graphs.

Theorem 1. (Tanna, 2013) Fan graph f_n is harmonious.

Proof. Consider an arbitrary fan graph f_n , having v_0 as its center vertex and v_1, v_2, \dots, v_n as the other n vertices of the graph. Construct a labeling function $f: V(G) \rightarrow \{0, 1, 2, \dots, 2n-2\}$ as follows

$$f(v_i) := \begin{cases} n + \frac{(i-1)}{2}, & \text{if } i \text{ is even} \\ \frac{n + (i-1)}{2}, & \text{otherwise.} \end{cases}$$

One can easily show that this function is injective, and its corresponding induced function is bijective. ■

Theorem 2. (Tanna, 2013) If $n \not\equiv 2 \pmod{4}$, then friendship graph F_n is harmonious. *Proof.* We consider the following two cases.

Case 1. Assume $n \equiv 0, 1 \pmod{4}$. Label the center vertex as 0. The set $\{1, 2, 3, \dots, 2n\}$ may be partitioned into n pairs (a_r, b_r) such that $b_r - a_r = r$ for $r = 1, 2, 3, \dots, n$ (Skolem, 1957). Then, we label the vertices of each triangle with $(0, r, n+a_r)$.

Case 2. Assume $n \equiv 3 \pmod{4}$. Again, label center vertex as 0. The set $\{1, 2, \dots, 2n-6\}$ can be partitioned into $n-3$ pairs (a_r, b_r) with $b_r - a_r = r+2$, for $r = 1, 2, \dots, n-3$. (Skolem, 1957). Then, we label the vertices of the triangles with $(0, 1, 3n-1)$, $(0, 2, 3n-6)$, $(0, 3n-2, 3n-3)$, and $(0, r+2, n+a_r)$ for $r=1, 2, \dots, n-3$.

In both cases, one can show that this vertex labeling is injective and its induced edge labeling is bijective. ■

Results and Discussion

Based on the construction of the labeling functions in the proof of the theorems presented in the preceding section, we present the following pseudocodes illustrating our algorithm for harmonious labeling of a fan graph f_n for any given number n of vertices and a friendship

graph F_n for any given number n of triangle components. Finally, we present some numerical results on harmonious labeling of certain fan and friendship graphs. We execute the algorithms presented below using the Fortran programming language (Markus, 2012) via Force application version 2.0.9p, and visualize the resulting harmoniously labeled graph using Gnuplot 5.0, a portable command-line driven graphing utility for Linux, OS/2, MS Windows, OSX, VMS, and many other platforms (Williams and Kelly, 2015).

We begin with a pseudocode for generating harmonious labeling of any given fan graph as follows.

Algorithm 1. Harmonious Labeling of Fan graph f_n .

input: number of non-central vertices N

output: vertex labels $V(N + 1)$; edge labels $E(2N - 1)$

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1 set  $V(N + 1) = 0$  (label central vertex)
2 while  $1 \leq i \leq N$  do
3   if  $i$  is even then
4      $V(i) = N + 0.5(i - 1)$  (label even vertices)
5   else
6      $V(i) = 0.5(N + i - 1)$  (label odd vertices)
7   end if
8   print ( $V(i)$ ) (procedure completed successfully)
9 end do
10 while  $1 \leq i \leq N$  do
11  $E(i) = (V(N + 1) + V(i)) \bmod(2N - 1)$  (label center-incident edges)
12   if  $i < N$  then
13      $E(N + i) = (V(i) + V(i + 1)) \bmod(2N - 1)$  (label path edges)
14   end if
15   print ( $E(i)$ ) (procedure completed successfully)
16 end do
17 stop
```

To illustrate a fan graph f_n with n path vertices, we partition a semicircle into $n-1$ sectors. The center of the semicircle is taken as the central vertex, while the endpoints of sector radii on the semicircle make up the remaining path vertices. We execute Algorithm 1 for $n=15$ and obtain a harmonious labeling of fan graph f_{15} (See Figure 3). Note that such fan graph visualization is not the only possible scheme. One can also consider n collinear points and connect them to a single non-collinear point.

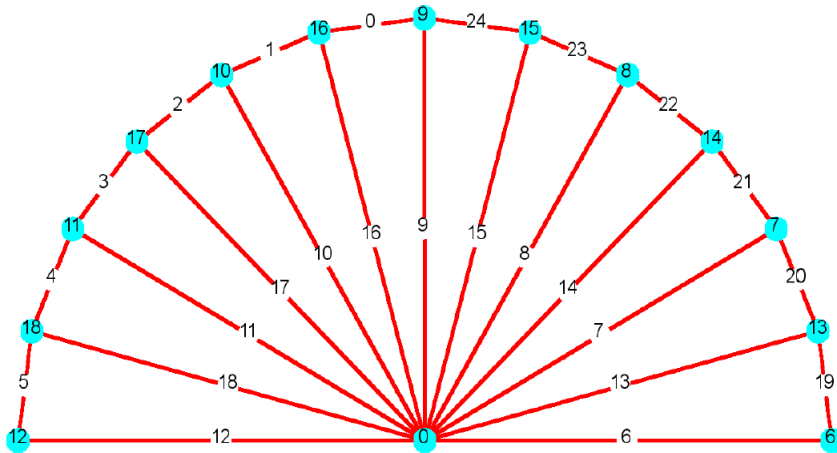


Figure 3. A harmonious fan graph f_{13} of order 13.

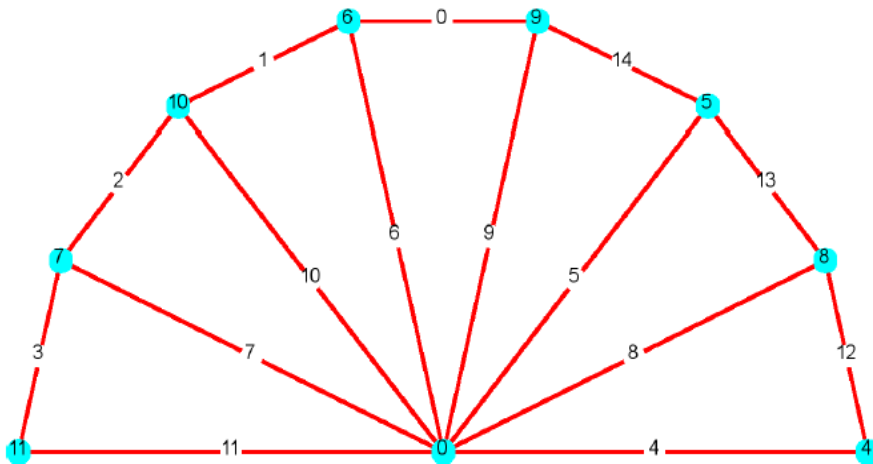


Figure 4. A harmonious fan graph f_8 of order 8.

Moreover, Figure 4 shows a harmonious labeling of fan graph where $n=8$. From the theorem of the fan graph, we labeled the path of the graph from $\{0, 1, 2, \dots, 2n\}$ following the given algorithm discussed in the previous section. The edges are also labeled from $\{0, 1, 2, \dots, 2n-2\}$ (modulo q), resulting in a harmonious labeling of the graph.

Next, we present a pseudocode for generating a harmonious labeling of any given friendship graph as follows.

Algorithm 2. Harmonious labeling of Friendship graph F_n

input: number of triangles N

output: vertex labels $V(2N+1)$; edge labels $E(3N)$

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1 if  $N \equiv 2 \pmod{4}$  then                                     (check  $N$ )
2   print("Friendship Graph is not Harmonious")
3 else
4   set  $V(2N+1)=0$                                          (label central vertex)
5   if  $N \equiv 0, 1 \pmod{4}$  then
6     initialize Dummy( $2N$ )=0
7     while Dummy  $\neq \{1, 2, \dots, 2n\}$  do
8       while  $1 \leq r \leq N$  do
9         find  $(a_r, b_r)$  with  $b_r - a_r = r$ 
10        Dummy( $r$ ) =  $a_r$ 
10        Dummy( $N+r$ ) =  $b_r$ 
11      end do
12    end do
13    while  $1 \leq r \leq N$  do
14       $V(2r-1) = r$                                        (label noncentral vertices)
15       $V(2r) = N + \text{Dummy}(r)$ 
16    end do
17  else
18     $V(2N) = 1$ 
19     $V(2N-1) = 3N-1$ 
20     $V(2N-2) = 2$ 
21     $V(2N-3) = 3N-6$ 
22     $V(2N-4) = 3N-2$ 
23     $V(2N-5) = 3N-3$ 
24    if  $N > 3$  then
25      initialize Dummy( $2N-6$ )=0
26      while Dummy  $\neq \{1, 2, \dots, 2N-6\}$  do
27        while  $1 \leq r \leq N-3$  do
28          find  $(a_r, b_r)$  with  $b_r - a_r = r+2$ 
29          Dummy( $r$ ) =  $a_r$ 
30          Dummy( $N+r$ ) =  $b_r$ 
31        end do
32      end do
33      while  $1 \leq r \leq N-3$  do
34         $V(2r-1) = r+2$                                    (label noncentral vertices)
35         $V(2r) = N + \text{Dummy}(r)$ 

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36         end do
37     end if
38 end if
39 print (V);                               (procedure completed successfully)
40 while 1≤i≤2N do
41     E(i)= (V(2N+1) + V(i)) mod (3N)      (label center-incident edges)
42 end do
43 while 1≤i≤N do
44     E(2N+i)= (V(2i-1) + V(2i)) mod (3N) (label remaining edges)
45 end do
46 print (E)                                (procedure completed successfully)
47 end if
48 stop

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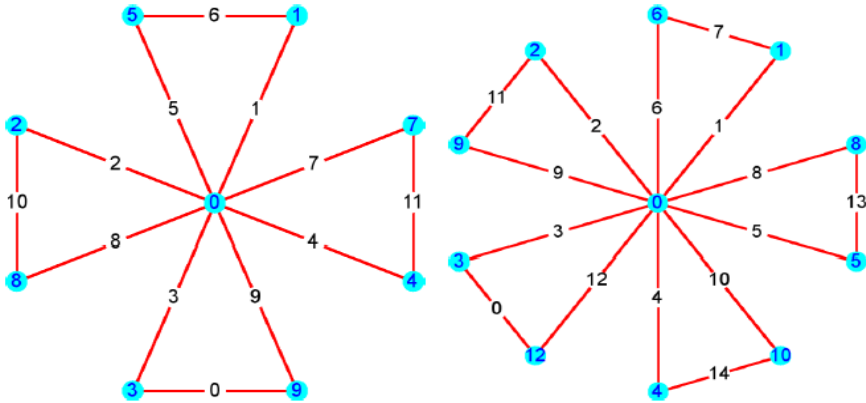


Figure 5. Harmonious friendship graphs F_4 (left) and F_5 (right).

For a friendship graph F_n , plotting its graph requires partitioning a circle into $2n$ sectors. The endpoints of radii, enclosing each sector, on the circle together with center make up the vertices of a friendship graph. We execute Algorithm 2 for different n values. First, we consider the case when $n \equiv 0, 1 \pmod{4}$. The partitioning scheme in our algorithm reveals six possible harmonious labeling for friendship graph F_4 ; and ten possible harmonious labeling for friendship graph F_5 . Resulting harmonious labeling via Algorithm 2 for friendship graphs F_4 and F_5 are shown in Figure 5.

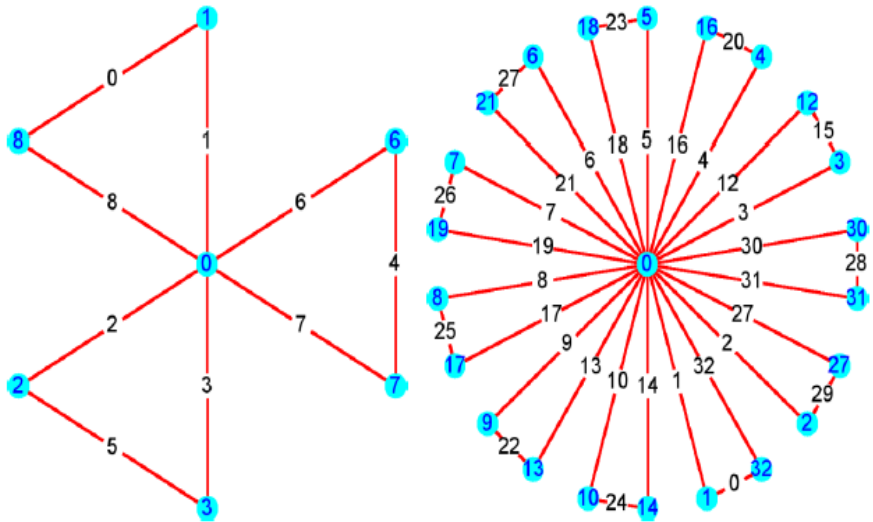


Figure 6. Harmonious friendship graphs F_3 (left) and F_{11} (right).

Lastly, we consider the case when $n \equiv 3 \pmod{4}$. Note that friendship graph F_3 does not require any partitioning. As a result, only one possible harmonious labeling is found by Algorithm 2, as shown in Figure 6. On other hand, for friendship graph F_{11} , our algorithm reveals 180 possible harmonious labeling – resulting from the partitioning scheme. One of these harmonious friendship graph F_{11} is shown in Figure 6.

Summary and Conclusion

We presented an algorithm for enumerating all possible harmonious labelings of a fan graph f_n for any given number n of vertices and a friendship graph F_n for any given number n of triangle components. The algorithms were coded and implemented via the Fortran programming language; and results were plotted via a command-line Gnuplot program. Other programming languages and plotting softwares can also be used in executing our algorithm. In constructing such algorithms, we utilized the results of Dushyant Tanna’s construction proof on “Harmonious Labeling of Certain Graphs” to write our algorithm

and generate numerical results. In generating a scheme for finding pairs (a_r, b_r) with $b_r - a_r = r + 2$, we listed all natural number pairs satisfying the condition and worked out the labeling thereafter. It was found that such scheme took the longest computational time to execute as the number n of triangle components in enumerating all harmonious labeling of a friendship graph F_n . For further study, we recommend that a new scheme should be put in place that is less computationally expensive. It would also be interesting to investigate other types of graphs as to whether they will admit harmonious labeling. In addition, the search for an alternative harmonious labeling function that is computationally cheaper as the size n increases is also a good research direction to take. Finally, one may also wish to generalize the search for harmonious labeling of any given graph, that is, without being restrictive to specific graphs.

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